AIAA Journal
Vol. 44, No. 3, March 2006

Technical Notes

TECHNICAL NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes should not exceed 2500 words (where a figure or table counts as 200 words). Following informal review by the Editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Static Analysis of Symmetric Angle-Ply Laminated Plates by Analytical Method

Yan Huang* and Duan-cai Yuan†
National University of Defense Technology, 410073 Changsha, People's Republic of China

I. Introduction

THE composite symmetric angle-ply laminated plates are anisotropic plates and have wide applications in aeronautic and astronautic engineering. There are several analytical and energy methods to study the bending problem of rectangular anisotropic plates.[1–5] Zhang and Yang[6] obtained a closed-form solution for a rectangular plate with arbitrary boundary conditions and uniformly distributed load using an exact complex series and a generalized algebraic polynomial. In the present study, we employ a double sine series to propose a general analytical solution for the static bending problem of an anisotropic plate with arbitrary boundary conditions and loading.

II. Solution of Partial Differential Equation

The partial differential equation for the bending deflection function of an anisotropic rectangular plate shown in Fig. 1 is[7]

$$D_{11}\frac{\partial^4 w}{\partial x^4}+4D_{16}\frac{\partial^4 w}{\partial x^3\partial y}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+4D_{26}\frac{\partial^4 w}{\partial x\partial y^3}+D_{22}\frac{\partial^4 w}{\partial y^4}=q \tag{1}$$

where w is the bending deflection; D_{11}, D_{12}, D_{16}, D_{22}, D_{26}, and D_{66} are flexural rigidity coefficients; and q is the external load per unit area. The solution of the homogeneous Eq. (1) will be of the form

$$w=e^{i\alpha x}e^{i\alpha' y}\qquad \text{or}\qquad w=e^{-i\alpha x}e^{-i\alpha' y}$$

Substituting the preceding expressions into Eq. (1) and letting $q=0$, we have

$$D_{11}\alpha^4+4D_{16}\alpha^3\alpha'+2(D_{12}+2D_{66})\alpha^2\alpha'^2+4D_{26}\alpha\alpha'^3+D_{22}\alpha'^4=0$$

The roots of α' in the preceding equation are $\alpha_{l1}\pm i\alpha_{l2}$, where $l=1,2$. Hence, we have

$$w=\exp[\pm i(\alpha x+\alpha_{l1}y)]\exp(\pm\alpha_{l2}y)$$

The preceding solutions will be rewritten in the form of trigonometric and hyperbolic function

$$w=[A\sin(\alpha x+\alpha_{l1}y+B\cos(\alpha x+\alpha_{l1}y)]\times(C\sinh\alpha_{l2}y+D\cosh\alpha_{l2}y) \tag{2}$$

Similarly, assuming that $w=e^{i\beta y}e^{i\beta' x}$, we have a similar solution to Eq. (2). Moreover, we have the algebraic polynomial solution

$$w_1=\alpha_{00}(1-\xi)(1-\eta)+\alpha_{01}(1-\xi)\eta+\alpha_{10}\xi(1-\eta)+\alpha_{11}\xi\eta \tag{3}$$

where

$$\xi=x/a,\qquad \eta=y/b$$

When $q\neq 0$ in Eq. (1), a particular solution of the double sine series form is assumed as

$$w_2=\sum_m\sum_n A_{mn}\sin\alpha x\sin\beta y \tag{4}$$

where

$$\alpha=m\pi/a,\qquad \beta=n\pi/b,\qquad m,n=1,2,3,\ldots$$

Substituting the preceding expression into Eq. (1) and letting

$$q=\sum_m\sum_n B_{mn}\sin\alpha x\sin\beta y$$

$$\cos\frac{i\pi x}{a}\cos\frac{i\pi y}{b}=\sum_m\sum_n C_{mn}\sin\alpha x\sin\beta y$$

where

$$B_{mn}=\frac{4}{ab}\int_0^a\int_0^b q\sin\alpha x\sin\beta y\,\mathrm{d}x\,\mathrm{d}y$$

$$C_{mn}=\frac{16mn}{\pi^2(m^2-i^2)(n^2-j^2)}$$

Where $i,\ j=1,2,3,\ldots$, only $i+m$ and $j+n$ are odd numbers; otherwise $C_{mn}=0$; then we have

$$A_{mn}\left[D_{11}\alpha^4+2(D_{12}+2D_{66})\alpha^2\beta^2+D_{22}\beta^4\right]-\sum_i\sum_j A_{ij}\left[4D_{16}\left(\frac{i\pi}{a}\right)^3\frac{j\pi}{b}+4D_{26}\frac{i\pi}{a}\left(\frac{j\pi}{b}\right)^3\right]\times\frac{16mn}{\pi^2(m^2-i^2)(n^2-j^2)}=B_{mn} \tag{5}$$

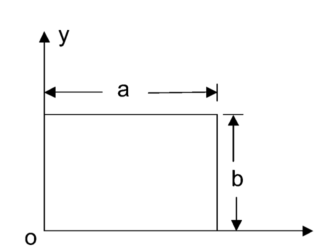


Fig. 1 Coordinates of plate.

Received 30 January 2005; revision received 27 August 2005; accepted for publication 28 August 2005.

*Professor, Institute of Aerospace and Material Engineering.
†Associate Professor, Institute of Aerospace and Material Engineering; ydc108@sohu.com.

When the load is uniformly distributed, $q = q_0 = \text{const}$, or when the concentrated load P acts at point $x = a_1$, $y = b_1$, we have

$$B_{mn} = 4q_0(1 - \cos m\pi)(1 - \cos n\pi)/mn\pi^2 \tag{6}$$

or

$$B_{mn} = 4P \sin\alpha a_1 \sin\beta b_1/ab \tag{7}$$

III. Establishment of a General Solution

In this Note, a general solution for solving the bending problem of anisotropic rectangular plates with arbitrary load and boundary conditions is suggested in the following form:

$$\begin{aligned} w = \sum_l \sum_m \Bigg\{ & A_{lm} \sin[\alpha(a-x) + \alpha_{l1}(b-y)] \frac{\sinh \alpha_{l2} y}{\sinh \alpha_{l2} b} \\ & + \frac{B_{lm} \sin(\alpha x + \alpha_{l1} y) \sinh \alpha_{l2}(b-y)}{\sinh \alpha_{l2} b} \Bigg\} \\ & + \sum_l \sum_n \Bigg\{ C_{ln} \sin[\beta(b-y) + \beta_{l1}(a-x)] \frac{\sinh \beta_{l2} x}{\sinh \beta_{l2} a} \\ & + \frac{D_{ln} \sin(\beta y + \beta_{l1} x) \sinh \beta_{l2}(a-x)}{\sinh \beta_{l2} a} \Bigg\} + w_1 + w_2 \end{aligned} \tag{8}$$

The first part of Eq. (8) satisfies any boundary condition along the edges $y = 0$ and $y = b$. The second part of Eq. (8) satisfies any boundary condition along $x = 0$ and $x = a$. Function w_1 must be added to satisfy the boundary conditions at the corner, and w_2 will be adopted to arbitrary load.

Equation (8) has $4m + 4n + 4$ integral constants that can be determined by boundary conditions. Along each edge there are two edge conditions: deflection or equivalent shearing force. Slope or bending moments should be equal to the given values along the edges, respectively. For every equation of edge conditions, all unsine functions need to be expressed in sine series. By orthogonality, $4m + 4n$ equations of edge conditions can be obtained. In addition, there are four corner conditions at all corners on deflection or reaction. Thus, the number of equations and integral constants are equal. Therefore, all of the integral constants can be determined. The functions in equations of boundary conditions expressed in sine series can be found in Ref. 8.

IV. Example

Assume for example, a rectangular plate with four edges free, supported at the center and with the load concentrated at the midpoint of four diagonals $(a/4, b/4)$, $(3a/4, b/4)$, $(a/4, 3b/4)$, and $(3a/4, 3b/4)$. The boundary conditions are

$$(M_x)_{x=0} = 0, \qquad (M_y)_{y=0} = 0, \qquad (M_x)_{x=a}, \qquad (M_y)_{y=b} = 0 \tag{9}$$

$$\begin{aligned} (V_x)_{x=0} = 0, \qquad (V_y)_{y=0} = 0 \\ (V_x)_{x=a} = 0, \qquad (V_y)_{y=b} = 0 \end{aligned} \tag{10}$$

$$R_{(0,0)} = 0, \qquad R_{(a,0)} = 0, \qquad R_{(0,b)} = 0, \qquad R_{(a,b)} = 0 \tag{11}$$

In addition, the deflection at the center is

$$w_{(a/2,b/2)} = 0 \tag{12}$$

in accordance with the condition of equilibrium. From Eq. (7), we have

$$\begin{aligned} B_{mn} = {} & P(1 - \cos m\pi - \cos n\pi + \cos m\pi \cos n\pi) \sin(m\pi/4) \\ & \times \sin(n\pi/4)/ab - 4P \sin(m\pi/2) \sin(n\pi/2)/ab \end{aligned} \tag{13}$$

It will be simpler to solve the bending problem if we make use of symmetric conditions in the deformation. For an anisotropic rectangular plate, when the load and boundaries are symmetric with respect to plate center,[6] then the boundary conditions in opposite edges and corners are identical, that is, the last two expressions of Eqs. (9–11) may not be used, and

$$A_{lm} = B_{lm}, \qquad C_{ln} = D_{ln}, \qquad a_{00} = a_{11}, \qquad a_{10} = a_{01}$$

For a laminated plate of symmetric angle-ply 45 deg/−45 deg/−45 deg/45 deg, it is easy to find that $D_{11} = D_{22}$, $D_{16} = D_{26}$, and $D_{12} + 2D_{66} > D_{11}$,

$$\alpha_{11}, \alpha_{21} = (\alpha/4)\left[-\lambda \pm \sqrt{2\sqrt{(u+2)^2 - 4\lambda^2} - 2(u+2) + \lambda^2}\right]$$

$$\alpha_{12}, \alpha_{22} =$$

$$(\alpha/4)\left[\sqrt{4u - 8 - \lambda^2} \mp \sqrt{2\sqrt{(u+2)^2 - 4\lambda^2} + 2(u+2) - \lambda^2}\right]$$

where $\lambda = 4D_{16}/D_{11}$ and $u = 2(D_{12} + 2D_{66})/D_{11}$. Similarly, β_{l1}, β_{l2} or γ_{l1}, γ_{l2} are the same as α_{l1}, α_{l2} if β or γ is used instead of α.

For a square plate $a = b$, the deformations are symmetric with diagonal $x = y$ and $x + y = a$. Thus, the remainder boundary conditions may not be used and $C_{ln} + A_{ln}$, Substituting Eqs. (3) and (4) into Eq. (8) and the first expression of Eqs. (9–12), according to Eqs. (5) and (13), we can solve A_{lm}, a_{00} and a_{01}.

For Poisson's ratio $\nu_1 = 0.25$ and Young's modulus $G_{12}/E_2 = 0.5$, for glass/epoxy composite $E_1/E_2 = 3$, $D_{11} = 0.1374E_2h^3$, $D_{12}/D_{11} = 0.3935$, $\lambda = 0.929$, and $u = 2.955$; and, for graphite/epoxy composite $E_1/E_2 = 40$, $D_{11} = 0.9076E_2h^3$, $D_{12}/D_{11} = 0.9082$, $\lambda = 2.6898$, and $u = 5.54077$. Taking m and n in terms of 8–32, we have the deflection w at the corner $x = 0$ and $y = 0$ shown in Table 1. It is known that the convergences are very good. When m and n are taken as 32 terms, we have the deflections along diagonal $x = y$ and $x + y = a$ shown in Table 2. The shapes are shown in Fig. 2. From Fig. 2, we see that $w(0, 0)$ is smaller than $w(a, 0)$. This is due to the flexural rigidity coefficient along diagonal $x = y$ being larger than that along diagonal $x + y = a$.

Moreover, we also calculated the bending deflection of square plate with layup angle [−45 deg/45 deg/45 deg/−45 deg]. For this angle-ply laminated plate, only D_{16} and D_{26} are negative numbers. The deflections along diagonal $x = y$ and $x + y = a$ are the same.

For comparison, we also calculate the same square plate with four edges simply supported, loaded uniformly, and made of a graphite/epoxy composite. Only the following boundary conditions are used:

$$(w)_{x=0} = 0, \qquad (M_x)_{x=0} = 0, \qquad w(0,0) = 0, \qquad w(a,0) = 0 \tag{14}$$

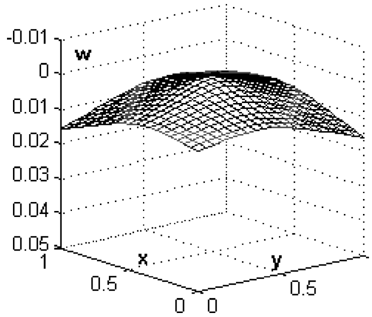
From the last two expressions, we have $a_{00} = a_{01} = 0$. From the first two expressions and attending Eq. (6), we can solve A_{lm}. When m and n are taken as 24 terms, we have the deflection at the center $w = 274 \times 10^{-5} qa^4/D_{11}$. The deflection contours map is shown in Fig. 3. This example is also solved by the finite elements method in Ref. 9. The resulting value and the contour map provided by Ref. 9 are in accordance with those in this Note.

Table 1 Convergence of deflections at the corner $w(0,0)(10^{-5}Pa^2/D_{11})$

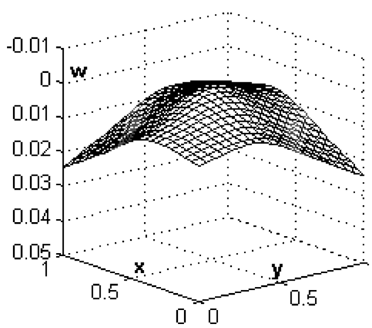
	m, n			
Plate	8	16	24	32
Glass/epoxy	872	915	940	941
Graphite/epoxy	1004	1036	1051	1060

Table 2 Deflection along the diagonal of a plate with four concentrated loads

		$w(10^{-5}Pa^2/D_{11})$				
Plate	x/a	0	0.1	0.2	0.3	0.4
Glass/epoxy	$y = x$	941	761	577	361	132
	$y = a - x$	1582	1241	908	550	198
Graphite/epoxy	$y = x$	1060	839	613	365	127
	$y = a - x$	2456	1914	1394	836	295



a) Glass/epoxy



b) Graphite/epoxy

Fig. 2 Deflection shapes of plates with concentrated load.

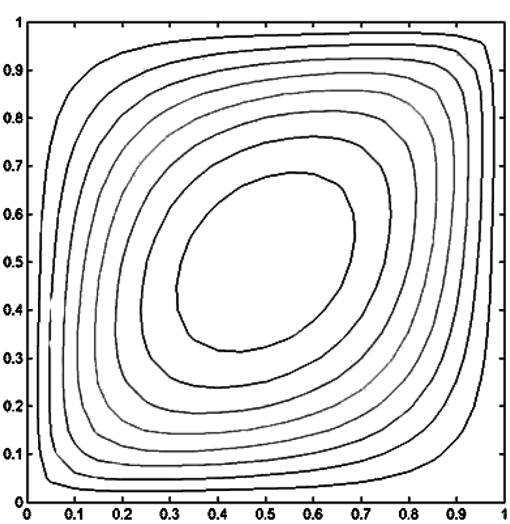


Fig. 3 Deflection contours map of plate.

V. Conclusions

In this Note, a general analytical solution is established. It can be used to solve the bending problem for an arbitrary load and with general boundaries. The procedure is as follows: 1) Give all of the boundary conditions of the four edges and the four corners. 2) Calculate A_{mn} by using Eq. (5). 3) Substitute the general solution described by Eq. (8) into all boundary conditions, and express all unsine functions in sine series to obtain the integral constant , deflection, bending moment, etc. 4) When the load and boundaries are symmetric with respect to plate center, the half-boundary condition may not be used, and, consequently, the half-integral constants are equal. 5) Moreover, for a square plate, when the material, load, and boundaries are symmetric with respect to the diagonal of the plate, the remainder half-boundary conditions may be not used, and the remainder half-integral constants are also equal, as in the example in this Note. 6) If we want to solve the bending moment at the corner, we must add all of the algebraic polynomial solutions in Ref. 6.

This theoretical analysis is simple. The method for calculation is easy and available in practical engineering.

Acknowledgment

The authors acknowledge support from the National Natural Science Foundation of the People's Republic of China (Grant 19872072).

References

[1] Yin, W. L., "General Solution of Anisotropic Laminated Plates," *Journal of Applied Mechanics*, Vol. 70, No. 4, 2003, pp. 496–504.

[2] Lim, G. T., and Reddy, J. N., "On Canonical Bending Relationships for Plates," *International Journal of Solids and Structures*, Vol. 40, No. 12, 2003, pp. 3039–3067.

[3] Hwu, C., "Stroh-Like Complex Variable Formalism for the Bending Theory of Anisotropic Laminated Plates," *Journal of Applied Mechanics*, Vol. 70, No. 5, 2003, pp. 696–703.

[4] Chen, P., and Nie, H., "Green's Function for Bending Problem of a Thin Anisotropic Plates with an Elliptic Hole," *Mechanics Research Communication*, Vol. 31, No. 3, 2004, pp. 311–320.

[5] Zhang, Y. X., and Kim, K. S., "Two Simple and Efficient Displacement Based Quartolatered Elements for the Analysis of Composite Laminated Plates," *International Journal for Numerical Methods in Engineering*, Vol. 61, No. 11, 2004, pp. 1771–1796.

[6] Zhang, C. Z., and Yang, G. S., "General Analytic Solutions for Transverse Bending Problem of Anisotropic Plate Structures," *Acta Mech Sinica*, Vol. 28, No. 4, 1996, pp. 429–440.

[7] Reddy, J. N., *Mechanics of Laminated Composite Plates and Shells*, CRC Press, Boca Raton, FL, 2004.

[8] Huang, Y., and Zhang, X. J., "General Analytical Solution of Transverse Vibration for Orthotropic Rectangular Thin Plates," *Journal of Marine Science and Application*, Vol. 1, No. 2, 2002, pp. 78–82.

[9] Cheng, G. D., "Discussion on Symmetry of Composite Square Plates," *Mechanics and Practice*, Vol. 7, No. 6, 1985, pp. 35–38.

A. Palazotto
Associate Editor